

Comment on “Arnowitt–Deser–Misner representation and
Hamiltonian analysis of covariant renormalizable gravity” by M.
Chaichian, M. Oksanen, A. Tureanu

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Abstract

The partial Hamiltonian analysis of the actions presented in the paper by M. Chaichian, M. Oksanen, A. Tureanu (*Eur. Phys. J. C* 71, 1657 (2011)) is incorrect; the true algebra of constraints differs from what they claim for their choice of momentum constraint. Our blind acceptance of the correctness of their constraint algebra led us to conclude, wrongly, that a few of the models presented by the authors (sharing the same constraint algebra) are not invariant under spatial diffeomorphism. We “proved” this by using Noether’s second theorem (see first version of the paper), but we then found a mistake in our calculations. The differential identity of spatial diffeomorphism is intact, therefore, their actions are invariant; but in this case, the spatial diffeomorphism gauge symmetry cannot be compatible with their algebra. We now explicitly demonstrate that the actual algebra of constraints is different, and briefly describe how it affects the generator and gauge transformations of the fields.

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The Hamiltonian analysis (partial) of the modified Hořava-type models was recently considered in [1] and [2, 3]. After elimination of the second-class constraints and the corresponding variables (three variables in [1] or one variable in [2, 3]) the total Hamiltonian¹ was obtained:

$$H_T = \dot{N}\pi + \int dx \dot{N}_i \pi^i + N \int dx H_0 + \int dx N_i H^i. \quad (1)$$

Note that the independent variables (and conjugate momenta) of these formulations are N, N_i, g_{pq} (π, π^i, π^{pq}) and scalar fields, e.g. $A(\pi_A)$ *et cetera*. The closure of the constraint algebra in the projectable case, $N = N(t)$, is claimed (see Eqs. (3.23)-(3.25) of [1], Eq. (43) of [2], and Eq. (101) of [3]) to be in the form:

$$\{\Phi_0, \Phi_0\} = 0, \quad (2)$$

$$\{\Phi_S(\xi_i), \Phi_0\} = 0, \quad (3)$$

$$\{\Phi_S(\xi_i), \Phi_S(\eta_j)\} = \Phi_S(\xi^j \partial_j \eta_i - \eta^j \partial_j \xi_i) \approx 0, \quad (4)$$

where

$$\Phi_0 = \int dx H_0, \quad \Phi_S(\xi_i) = \int dx \xi_i H^i. \quad (5)$$

Equations (2)-(4) are similar to those presented by Hořava [4], which are based on the known results for General Relativity (GR) in ADM variables with the additional projectability condition, $N = N(t)$; but in the Hořava paper [4], the constraint algebra (taken from the Hamiltonian formulation of GR in ADM variables) was written for a different choice of shift variable N^k , and its conjugate momentum π_k (primary constraint), and its time development leads to the momentum constraint H_k , which is related to H^i of [1–3] by

$$H_k = g_{ik} H^i. \quad (6)$$

¹ Instead of using undetermined Lagrange multipliers in front of the primary constraints in (1), we write the undetermined velocities as they appear in the Legendre transformation. In the case under consideration this difference is not important, but for some formulations it is crucial for the restoration of the gauge transformations of all fields.

This relationship (see Eq. (6)) is different from that for ordinary field theory where the indices are raised and lowered by the Minkowski tensor with no effect on the Poisson Brackets (PBs). Further, ξ_i in [1–3] is a test function, which is just a different form of presentation of the PBs of two constraints (e.g. see section 3 of [6]), that avoids the appearance of delta functions and their derivatives. The test functions are assumed to have a zero PB with all variables presented in the constraints, but this property would be destroyed if one were to use $g_{ik}\xi^k$ instead of ξ_i .

The Hořava algebra is

$$\{\Phi_0, \Phi_0\} = 0, \quad \{\Phi_S(\xi^i), \Phi_0\} = 0, \quad (7)$$

$$\{\Phi_S(\xi^i), \Phi_S(\eta^j)\} = \Phi_S(\xi^j \partial_j \eta^i - \eta^j \partial_j \xi^i) \approx 0, \quad (8)$$

where

$$\Phi_0 = \int dx H_0, \quad \Phi_S(\xi^k) = \int dx \xi^k H_k.$$

It is well known that if some constraints are linear combinations of others, with field-dependent coefficients (fields that are canonically conjugate to variables presented in constraints, i.e. as (6)), then the constraint algebra cannot be the same. Such a change cannot affect closure of the algebra, but the form of closure must be different. Therefore, from a comparison of the two algebras, one ought to conclude that spatial diffeomorphism is not the gauge symmetry of the models in [1–3].

The first version of this comment was based on this observation, and we had decided to show, using Noether's second theorem, that there is no spatial diffeomorphism gauge symmetry for actions that lead to (2)-(4). But after the previous version appeared on arXiv, we found that the contribution that we claimed destroys the Differential Identity (DI) of spatial diffeomorphism can in fact be compensated. All our attempts to rescue our conclusion and to find another term did not succeed. Therefore, we returned to the Hamiltonian formulation of [1], since the only way to reconcile our corrected Lagrangian analysis with the Hamiltonian one was to investigate the possibility that the algebra of constraints (2)-(4) stated in [1–3] is incorrect. Both approaches, Lagrangian and Hamiltonian, must give the

same result; and if the DI of spatial diffeomorphism is valid, then spatial diffeomorphism is the gauge symmetry, but in this case algebra (2)-(4) must be different.

To check this algebra, all variables that correspond to secondary constraints must be eliminated; this process makes the Hamiltonian analysis of [1] complicated, and unfortunately the Hamiltonian constraint was not written explicitly in the paper. Therefore we switch to another simpler model of the authors, discussed in [2], in which only one field was eliminated by solving a simple equation. For action Eq. (27) [2], the total Hamiltonian (1) has the following explicit form of the Hamiltonian and momentum constraints (see Eq. (34) of [2] or Eq. (96) of [3]):

$$H_0 = \frac{1}{\sqrt{g}} \left[\frac{1}{B} \left(g_{ik} g_{jl} \pi^{ij} \pi^{kl} - \frac{1}{3} g_{ij} g_{kl} \pi^{ij} \pi^{kl} \right) - \frac{1}{3\mu} g_{pq} \pi^{pq} \pi_B - \frac{1-3\lambda}{12\mu^2} B \pi_B^2 \right] \quad (9)$$

$$+ \sqrt{g} \left[B (E^{ij} G_{ijkl} E^{kl} + A) - F(A) + 2\mu g^{ij} \nabla_i \nabla_j B \right],$$

$$H^i = -2\partial_j \pi^{ij} - g^{ij} (2\partial_k g_{jl} - \partial_j g_{kl}) \pi^{kl} + g^{ij} \partial_j B \pi_B \quad (10)$$

where, unlike the complicated Eqs. (3.19)-(3.20) of [1], one has to substitute $A = \tilde{A}(B)$, which results from the elimination of the second-class constraints (see section 2.4 of [5])².

Because the expression for the Hamiltonian constraint is complicated, even for this model, we begin with a calculation of (3) for a particular term (second in (9)),

$$\left\{ (-2\partial_j \pi^{ij} - g^{ij} (2\partial_k g_{jl} - \partial_j g_{kl}) \pi^{kl} + g^{ij} \partial_j B \pi_B) (\mathbf{x}), \int d^3 \mathbf{y} \left(-\frac{1}{3\mu} \right) \left(\frac{1}{\sqrt{g}} g_{pq} \pi^{pq} \pi_B \right) (\mathbf{y}) \right\}, \quad (11)$$

which obviously produces non-zero contributions. The simplest non-zero contribution of (11) is

$$-\frac{1}{3\mu} (\partial_j B \pi_B) (\mathbf{x}) \int d^3 \mathbf{y} \left[\{ g^{ij} (\mathbf{x}), \pi^{pq} (\mathbf{y}) \} \left(\pi_B \frac{1}{\sqrt{g}} g_{pq} \right) (\mathbf{y}) \right] = \frac{1}{3\mu} \frac{1}{\sqrt{g}} g^{ij} \partial_j B \pi_B^2 (\mathbf{x}), \quad (12)$$

which is proportional to π_B^2 .

² To simplify the notation in this comment we omit the superscript ⁽³⁾, i.e. $g_{km}^{(3)} = g_{km}$, $\sqrt{g^{(3)}} = \sqrt{g}$.

All other contributions that originate from (11) are linear in momentum, π_B , and cannot compensate the term in the right-hand side of (12). There is only one term in the Hamiltonian constraint (third term of (9)) which is a possible source of contribution (12), but it enters with a different numerical coefficient. Contribution (12) can be compensated, in principle, but it would unavoidably lead to a restriction on the coefficient (i.e. a particular relationship between μ and λ). But the PB of the momentum constraint with the third term of (9) is

$$\left\{ H^i(\mathbf{x}), \int d^3\mathbf{y} \left(-\frac{1-3\lambda}{12\mu^2} \right) \left(\frac{1}{\sqrt{g}} B \pi_B^2 \right) (\mathbf{y}) \right\} = 0, \quad (13)$$

and compensation of (12) is impossible, even with a restriction on the parameters; therefore, the claim made by the authors that PB (3) is zero for their choice of variables (N_i and the corresponding momentum constraint H^i (10)) is incorrect.

The structure of contribution (12) precludes finding a result that is proportional to the Hamiltonian constraint, and to have closure on the secondary constraint, only a proportionality to the momentum constraint can be expected. Indeed, the calculation of all contributions of (11) provides such a result

$$\left\{ H^i(\mathbf{x}), \int d^3\mathbf{y} \left(-\frac{1}{3\mu} \right) \left(\frac{1}{\sqrt{g}} g_{pq} \pi^{pq} \pi_B \right) \right\} = \frac{1}{3\mu} \frac{\pi_B}{\sqrt{g}} H^i. \quad (14)$$

Of course all terms of (9) should be used, but even if there is only one non-zero contribution (14) and there is closure, the generator built as in [7] for GR in ADM variables (all first-class constraints and their algebra are needed) will be modified, and the transformations will be different. In fact, we found additional non-zero contributions for PB (3), and all of them are proportional to the momentum constraint (10). The PB among two momentum constraints (4) is also different from that claimed in the papers [1–3, 5].

But because the constraint algebra given in [1–3, 5] is incorrect, it is natural to expect that when the standard choice of shift function, N^k , with the primary constraint π_k , and the momentum constraint

$$H_k = -g_{ki} 2\partial_j \pi^{ij} - (2\partial_m g_{kl} - \partial_k g_{ml}) \pi^{ml} + \partial_k B \pi_B, \quad (15)$$

are used, the Hořava algebra (7)-(8) would follow. We also note that calculations with (15) are much simpler to perform than with (10). The PBs for the parts we considered (i.e. (13)

and (14)) for the constraint (15) are just zero and the rest of the terms in (9) also give zero. The PB of two momentum constraints (15) is also very easy to calculate by using the known result for GR in ADM variables, due to the decoupling of the additional fields

$$H_k = H_k^{GR} + \partial_k B \pi_B, \quad \{H_k^{GR}, \partial_k B \pi_B\} = 0.$$

So for the standard choice (N^k, π_k, H_k) , the Hořava algebra (7)-(8) follows.

To restore the transformations of all fields, as in GR, the generator is built from all of the first-class constraints, and it is at this point that the entire constraint algebra enters the game. For GR in ADM variables the generator is known (see [7] Eq. (29)), and it needs slight modification (taking into account the projectability condition)

$$G(\xi^k) = \int d^3x \left[\dot{\xi}^k \pi_k + \xi^k (H_k + \partial_k N^j \pi_j + \partial_j (N^j \pi_k)) \right]. \quad (16)$$

This is the generator of the gauge transformations that corresponds to the primary first-class constraint π^k and gives transformations of *all* fields³

$$\delta field = \{G, field\} \quad (17)$$

With generator (16), we find the following gauge transformations:

$$\delta N = 0,$$

$$\delta N^i = -\dot{\xi}^i - \xi^k \partial_k N^i + \partial_j \xi^k N^j, \quad (18)$$

$$\delta g_{pq} = -g_{qm} \partial_p \xi^m - g_{pm} \partial_q \xi^m - \xi^j \partial_j g_{pq},$$

$$\delta B = -\xi^i \partial_i B, \quad (19)$$

$$\delta A = -\xi^i \partial_i A. \quad (20)$$

³ Here, a different convention is followed. To us, this choice seems more natural, and it corresponds to [2]; but in Castellani's paper [7] a different order is used, and this is why he has a minus sign on the right-hand side of his Eq. (29), compared with our (16).

For the field A (not presented in generator (16)) the transformation is obtained as follows: one must return to the eliminated second-class constraints (see section 2.4 of [5]), which were solved for A as some function of B alone; because of this condition, the explicit form is irrelevant. Let us consider $A = \tilde{A}(B)$ – the transformation is $\delta A = \delta \tilde{A}(B) = \frac{\delta \tilde{A}(B)}{\delta B} \delta B = -\frac{\delta \tilde{A}(B)}{\delta B} \xi^i \partial_i B$, and the derivatives of the same equation are $\partial_i A = \partial_i \tilde{A}(B) = \frac{\delta \tilde{A}(B)}{\delta B} \partial_i B$; the combination of these two results leads to (20). Transformation (18) is the same as that given in Eq. (41) of [8] for $\partial_k N = 0$ (projectable case) and $\xi^0 = 0$.

Note: to derive the generator (16) (see [7]) the PB of H_k with the total Hamiltonian is needed, or $\{H_i, \int dy N^i H_i\}$ and $\{H_i, \int dy H_0\}$. These results are encoded in the constraint algebra, in particular, using (8)

$$\{\Phi_S(\xi^i), \Phi_S(\eta^j)\} = \left\{ \int dx \xi^k H_k, \int dy \eta^i H_i \right\} = \int dx (\xi^j \partial_j \eta^i - \eta^j \partial_j \xi^i) H_i, \quad (21)$$

$$\int dx \xi^k \left\{ H_k, \int dy \eta^i H_i \right\} = \int dx \xi^k (\partial_k \eta^j H_j + \partial_j (\eta^j H_k)); \quad (22)$$

and by putting $\eta^j = N^j$, we then obtain

$$\left\{ H_k, \int dy N^i H_i \right\} = \partial_k N^j H_j + \partial_j (N^j H_k).$$

Let us build the generator for the choice of variables in [1–3, 5] under the assumption that the constraint algebra (2)-(4) presented in papers [1–3, 5] is correct. Calculations for (4) (similar to (21)-(22)) for the choice of N_i , lead to the PB

$$\left\{ H^k, \int dy N_i H^i \right\} = g^{jk} \partial_j N_i H^i + \partial_j (g^{ji} N_i H^k)$$

and to the generator

$$G(\xi_k) = \int d^3x \left[\dot{\xi}_k \pi^k + \xi_k (H^k + g^{mk} \partial_m N_j \pi^j + \partial_j (N^j \pi^k)) \right]. \quad (23)$$

Using (17) with generator (23), one obtains the gauge transformations:

$$\delta N = 0, \quad (24)$$

$$\delta N_i = -\dot{\xi}^m g_{im} - \xi^m \dot{g}_{im} - \xi^m \partial_m N_i + \partial_j (g^{jm} N_m g_{ni} \xi^n), \quad (25)$$

$$\delta g_{pq} = -g_{qm}\partial_p\xi^m - g_{pm}\partial_q\xi^m - \xi^j\partial_j g_{pq}, \quad (26)$$

$$\delta B = -\xi^i\partial_i B.$$

Note that after the calculation (ξ_k in (23)) one has to substitute $\xi_k = g_{ki}\xi^i$ (for the standard choice of variables and generator (16), ξ^i automatically appears). Transformation (25) is not a spatial diffeomorphism transformation, and this fact might explain why the Hamiltonian formulation in [1–3, 5] is incomplete; and why the transformations for lapse and shift were not reported. The situation is worse if the true closure for (3)-(4) is used to construct the corresponding generator. Generator (23) becomes much more complicated, e.g. term (14) leads to a contribution to the generator,

$$G'(\xi_k) = \int d^3x \left[\dot{\xi}_k \pi^k + \xi_k \left(H^k + \frac{1}{3\mu} \frac{\pi_B}{\sqrt{g}} \pi^i + F(\pi^i) \right) \right]. \quad (27)$$

Just this one extra term, which is explicitly written in Eq. (27), modifies the transformations of fields N_i , B , and π^{pq} ; and the result not only differs from spatial diffeomorphism, but the primary constraint, π^i , will appear in the transformations of B and π^{pq} (see (17)), making it impossible to find the gauge transformations in configurational space.

This failure is related to the field-parametrisation dependence of the Dirac method [9]. For a different parametrisation (choice of independent variables and especially “primary” variables, i.e. those for which the conjugate momenta are primary first-class constraints) a different gauge invariance follows. For example, the Hamiltonian formulation of the Einstein-Hilbert action in the original variables, metric, leads to full diffeomorphism in the formulation of either Pirani-Schild-Skinner (PSS) [10, 11] or Dirac [12, 13], but in ADM variables the transformations are different [13] (for connection with Lagrangian methods see [8, 14]). The difference in these gauge transformations is caused by the non-canonical relationship of the two sets of variables; but for PSS and Dirac, the result is the same since they are canonically related and have the same algebra of constraints [15]. Some mechanical (finite dimensional) models are known, in which one cannot restore the gauge invariance in the Hamiltonian formulation. These cases are often presented as so-called counterexamples for the Dirac conjecture [16], but all such counterexamples can be explained by a change of field parametrisation. The choice of variables in [1–3, 5] provides, to the best of our knowledge,

the first field-theoretical “counterexample”. Is this behaviour specific to models with extra scalar fields and the projectability condition, or can GR in ADM variables also exhibit problems if one were to choose the shift in the form N_i (we are unaware of a Hamiltonian analysis for this choice of variables)? It is difficult to answer these questions because a new constraint algebra must be calculated, and it is much more complicated than it would be with the standard choice of variables. For example, even all structure “constants” of such an algebra are field dependent. Because the change of variables $N^i \rightarrow N_i$ is not canonical (the algebra is different), the known symmetry for the ADM formulation cannot be expected, even if the transformations for configurational variables could be restored.

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